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High-Frequency Expansion of Relativistic Classical Plasma Dielectric Tensor

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A high-frequency sum-rule expansion is derived for the transverse elements of the relativistic classical plasma dielectric tensor in an isotropic system. The relativistic results are different from the nonrelativistic ones by a factor of $\langle\gamma^{-1}(1-v^2/3c^2)\rangle$ for $\Omega_2^{\mu\nu}(\mathbf{k})$ and longitudinal plasmon and transverse photon frequencies, and by $\langle\gamma^{-2}(1-2v^2/3c^2)\rangle$ for $\Omega_4^{\mu\nu}(\mathbf{k})$.

1. INTRODUCTION

High-frequency sum-rule expansions of the full response tensor of nonrelativistic classical and quantum plasmas both in the absence and presence of an external magnetic field are known (Kalman and Genga, 1986; Genga, 1988*a*, *c*; *b*, in press). However, the relativistic plasma case has received no attention. It is known that in an isotropic system the dielectric tensor has two independent elements, the longitudinal and transverse (with respect to the wave vector k) elements. Therefore, in order to study the high-frequency behavior of the full dielectric tensor, one has to analyze the transverse element.

In this work I study the high-frequency behavior of the full dielectric tensor in an isotropic situation up to order ω^{-4}. The method of derivation relies on the Hamiltonian formalism (Kalman and Genga, 1986; Genga, 1988*a*). As in the nonrelativistic case (Kalman and Genga, 1986; Genga, 1988*a*), one also enlarges the Hamiltonian so as to include the photon degrees of freedom; this allows the description of the transverse interaction. As a result of this, it is known (Kalman and Genga, 1986; Genga, 1988*a*) that in addition to particle contributions to the sum-rule coefficients, the

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photon gas coexistent with the high-temperature plasma generates its own contribution. The exact evaluation of this contribution is hindered by two conditions (Kalman and Genga, 1986; Genga, 1988*a*): First, the classical ultraviolet divergence requires that even within the framework of a classical theory one describes the photons via the quantum Bose–Einstein distribution. Second, the equilibrium description implies that one single temperature exists for the combined photon system. Such an equilibrium exists in the astrophysical situation. Thus, an ad hoc approximation described in Section 2 is used to decouple the photons from the particle system.

In Section 2, I review the general relationships between the external or current–current response function sum-rule coefficients and those of the dielectric tensor. Then I calculate the exact ω^{-4} sum-rule coefficient for the transverse element. The long-wavelength limit of the result is calculated in Section 3; its possible implications for the dispersion relation of transverse plasma modes is also discussed. The results in Section 2 are obtained by using the same procedure as that given for the nonrelativistic classical magnetic-field-free case (Kalman and Genga, 1986, Appendix).

2. TRANSVERSE SUM RULES

The full dielectric tensor $\varepsilon^{\mu\nu}(\mathbf{k}\omega)$ and the full polarizability tensor $\alpha^{\mu\nu}(\mathbf{k}\omega)$ are expressible in terms of the corresponding "external" quantity $\hat{\alpha}^{\mu\nu}(\mathbf{k}\omega)$ as

$$\begin{aligned} \boldsymbol{\alpha} &= \hat{\boldsymbol{\alpha}}(\boldsymbol{\Delta}-\boldsymbol{\alpha})^{-1}\boldsymbol{\Delta} \\ \boldsymbol{\Delta} &= \mathbf{11} - n^2\mathbf{T} \\ n &= kc/\omega \\ \mathbf{T} &= \mathbf{11} - \mathbf{kk}/k^2 \end{aligned} \tag{1}$$

and $\hat{\alpha}^{\mu\nu}(\mathbf{k}\omega)$ possesses the well-known high-frequency sum-rule expansion

$$\hat{\boldsymbol{\alpha}}^{H'}(\mathbf{k}\omega) = -\sum_{\substack{l=1 \\ l\ \mathrm{odd}}}^{\infty} \frac{\boldsymbol{\Omega}_{l+1}(\mathbf{k})}{\omega^{l+1}} \tag{2}$$

$$\boldsymbol{\alpha}^{H''}(\mathbf{k}\omega) = -\sum_{\substack{l=2 \\ l\ \mathrm{even}}}^{\infty} \frac{\hat{\boldsymbol{\Omega}}_{l+1}(\mathbf{k})}{\omega^{l+1}} \tag{3}$$

where the superscript H stands for "Hermitian part of" and prime and double prime denote "real part of" and "imaginary part of," respectively. The $\hat{\boldsymbol{\Omega}}$ coefficients are evaluated from the relation (Kalman and Genga, 1986; Genga, 1988*a*)

$$\hat{\Omega}^{\mu\nu}_{l+1}(\mathbf{k}) = \frac{4\pi e^2}{v}\beta\left(i\frac{d}{d\tau}\right)^{l-1}\langle \mathbf{j}^{\mu}_{\mathbf{k}}(\tau)\mathbf{j}_{\mathbf{k}}(0)\rangle\big|_{\tau=0} \tag{(4)}$$

where

$$\mathbf{j}^{\mu}_{\mathbf{k}} = \sum_i v^{\mu}_i \exp(-i\mathbf{k}\cdot\mathbf{x}_i) \tag{5}$$

The high-frequency expansion of $\boldsymbol{\alpha}(k\omega)$ is similar to that of $\hat{\boldsymbol{\alpha}}(k\omega)$ as given by (2) and (3), with $\boldsymbol{\Omega}_{l+1}(k)$ replacing the corresponding $\hat{\boldsymbol{\Omega}}_{l+1}(k)-s$.

As mentioned in the introduction, the Hamiltonian which takes into account the description of the interaction of the plasma with the transverse electromagnetic field must include the photon degrees of freedom. Therefore we have

$$H = \sum_{i=1}^{N} \gamma_i mc^2 + \tfrac{1}{2}\sum_{i\neq j} V(|\mathbf{x}_i - \mathbf{x}_j|) + \tfrac{1}{2}\sum_{\mathbf{q}} (\tilde{e}_{\mathbf{q}}\bar{\mathbf{e}}_{\mathbf{q}} + q^2c^2\bar{a}_{\mathbf{q}}\bar{\mathbf{a}}_{\mathbf{q}}) \tag{6}$$

where

$$\gamma_i = \left(1 - \frac{v_i^2}{c^2}\right)^{-1/2}$$

$$\mathbf{V}_i = \frac{[\bar{P}_i + i\Lambda\sum_{\mathbf{q}}\bar{a}_{\mathbf{q}}\exp(i\mathbf{q}\cdot\mathbf{x}_i)]/m}{\{1+(1/m^2c^2)[\bar{P}_i + i\Lambda\sum_{\mathbf{q}}\bar{a}_{\mathbf{q}}\exp(i\mathbf{q}\cdot\mathbf{x}_i)]^2\}^{1/2}} \tag{7}$$

$$\Lambda = ie\left(\frac{4\pi}{V}\right)^{1/2}$$

Since we are considering the magnetic-field-free case, the system is isotropic and $\hat{\boldsymbol{\alpha}}$ is diagonal; thus, only $\hat{\boldsymbol{\Omega}}_2$ and $\hat{\boldsymbol{\Omega}}_4$ survive. Further, in a coordinate system in which $\mathbf{k} = (0, 0, \mathbf{k})$, one is left only with Ω^{22}_n and Ω^{33}_n elements.

The first moment yields

$$\begin{aligned}\hat{\Omega}^{\mu\nu}_2(\mathbf{k}) &= \frac{4\pi e^2}{V}\beta_p\langle j^{\mu}_{\mathbf{k}}(0)j^{\nu}_{-\mathbf{k}}(0)\rangle \\ &= \omega_p^2\left\langle\gamma^{-1}\left(\delta^{\mu\nu} - \frac{V^{\mu}V^{\nu}}{c^2}\right)\right\rangle \\ &= \omega_p^2\left\langle\gamma^{-1}\left(1 - \frac{V^2}{3c^2}\right)\right\rangle\delta^{\mu\nu}\end{aligned} \tag{8}$$

The third moment leads to

$$\hat{\Omega}_4^{\mu\nu}(\mathbf{k}) = \frac{4\pi e^2}{V}\,\beta\langle \dot{j}_{\mathbf{k}}^{\mu\nu}(0)\dot{\mathbf{j}}_k(0)\rangle$$

$$= \frac{4\pi e^2}{V}\,\beta \sum_{ij} \langle (\dot{V}_i^\mu \dot{V}_i^\nu + k^\alpha k^\delta V_i^\alpha V_i^\alpha V_i^\mu \dot{V}_j^\nu - ik^\alpha V_i^\alpha V_j^\nu$$
$$+ ik^\alpha \dot{V}_i^\mu V_j^\delta V_j^\nu) \exp[-i\mathbf{k}\cdot(\mathbf{x}_i - \mathbf{x}_j)]\rangle, \tag{9}$$

where $\dot{V}_i^\mu$ is the acceleration of the ith particle in the μ direction given by

$$\dot{V}_i^\mu = \frac{\gamma_i^{-1}}{m}\left(\delta^{\mu\alpha} - \frac{V_i^\mu V_i^\alpha}{c^2}\right)\left\{-\frac{\partial\Phi}{\partial x_1^\alpha} - i\Lambda \sum_{\mathbf{q}} e_{\mathbf{q}}^\alpha \exp(i\mathbf{q}\cdot\mathbf{x}_i)\right.$$
$$\left. + \Lambda \sum_{\mathbf{q}} [\mathbf{V}_i \times (\mathbf{q}\times \bar{a}_{\mathbf{q}})]^\alpha \exp(i\mathbf{q}\cdot\mathbf{x}_j)\right\} \tag{10}$$

with

$$\frac{\partial\Phi}{\partial x_1^\alpha} = \frac{\partial H}{\partial x_1^\alpha} - \frac{\partial K}{\partial x_i^\alpha}$$
$$K = \sum_i \gamma_i mc^2 \tag{11}$$
$$\Phi = \tfrac{1}{2}\sum_{i=j} V(|\mathbf{x}_i - \mathbf{x}_j|)$$

As in the nonrelativistic classical plasma case (Kalman and Genga, 1986; Genga, 1988*a*), the presence of the photon degrees of freedom $a_{\mathbf{q}}^\mu$, $e_{\mathbf{q}}^\mu$ gives rise to averages of field coordinates of the form $\langle a_{\mathbf{q}}^\mu \mathbf{a}_{\mathbf{q}}^\mu\rangle$ and $\langle e_{\mathbf{q}}^\mu \mathbf{e}_{\mathbf{q}}^\mu\rangle$, which in strict thermal equilibrium are expressible in terms of inverse particle temperature β_p and radiation temperature β_r, respectively. These averages have to be calculated quantum mechanically even in the framework of a classical theory such as the one under consideration. Introducing

$$c_{\mathbf{q}}^i = \frac{1}{\sqrt{2}}\left(\omega_{\mathbf{q}}^{1/2} a_{\mathbf{q}}^\mu + \frac{i}{\omega_{\mathbf{q}}^{1/2}} e_{\mathbf{q}}^\mu\right)\varepsilon_{\mathbf{q}}^{\mu i}$$
$$\omega_q = qc \tag{12}$$

as a new set of coordinates with the polarization vectors $\varepsilon_q^{\mu i}$, and

$$n_q^i = c_{\mathbf{q}}^{i*} c_{\mathbf{q}}^i \tag{13}$$

identified as the equivalent of the photon number operator, then averages are evaluated by setting (Kalman and Genga, 1986; Genga, 1988*a*)

$$\langle n_{\mathbf{q}}^i\rangle = \frac{1}{\exp(\beta_r \hbar\omega_{\mathbf{q}}) - 1}$$

Thus, equation (9) leads to

$$\hat{\Omega}_4^{\mu\nu}(\mathbf{k}) = \omega_p^4 \Big\langle \gamma^{-2}\left(1-\frac{2V^2}{3c^2}\right)\Big[\delta^{\mu\nu} + \frac{\beta_p}{\beta_r} T_{\mathbf{k}}^{\mu\nu}[f(x_{\mathbf{k}})-1] + \frac{k^2}{\kappa^2}(3L_{\mathbf{k}}^{\mu\nu} + T_{\mathbf{k}}^{\mu\nu}) + \frac{1}{N}\sum_{\mathbf{q}} L_{\mathbf{q}}^{\mu\nu}(S_{\mathbf{k}-\mathbf{q}} - S_{\mathbf{q}}) + \frac{\beta_p}{\beta_r}\frac{1}{N}\sum_{\mathbf{q}} T_{\mathbf{q}}^{\mu\nu} S_{\mathbf{k}-\mathbf{q}} f(x_{\mathbf{q}})\Big]\Big\rangle \tag{14}$$

where

$$\begin{aligned} L_{\mathbf{k}}^{\mu\nu} &= \frac{k^\mu k^\nu}{k^2} \\ T_{\mathbf{k}}^{\mu\nu} &= \delta^{\mu\nu} - \frac{k^\nu k^\nu}{k^2} \\ x_{\mathbf{k}} &= \hbar\omega_{\mathbf{k}}\beta_r \\ f(x) &= \frac{x}{e^x - 1} \end{aligned} \tag{15}$$

Equation (14) is obtained by arguing as before (Kalman and Genga, 1986; Genga, 1988*a*) that in the limit $k \to 0$ the distinction between particle (longitudinal) and radiation (transverse) temperatures is meaningless; hence, β_r is treated as a k-independent quantity, such that $\beta_r(k\to 0) = \beta_p$, while $\beta_r(k \neq 0)$ is unaffected by this condition.

3. LONG-WAVELENGTH LIMIT

The elements of the frequency moments in the long-wavelength ($k \to 0$) limit are as follows:

$$\begin{aligned} \hat{\Omega}_2^{11}(\mathbf{k}) &= \hat{\Omega}_2^{22}(k) = \hat{\Omega}_2^{33} = \omega_p^2 \left\langle \gamma^{-1}\left(1 - \frac{V^2}{3c^2}\right)\right\rangle \\ \hat{\Omega}_4^{11}(\mathbf{k}) &= \hat{\Omega}_4^{22}(\mathbf{k}) = \omega_p^4 \left\{ \gamma^{-2}\left(1-\frac{2V^2}{3c^2}\right)\right\rangle \\ &\quad \times \left\{1 + \frac{k^2}{\kappa^2}(1 - \tfrac{2}{15}\beta_p E_{\text{corr}}) + \frac{\beta_p}{\beta_r^4 n\hbar^3 c^3}\right. \\ &\quad \times \left.\left[\frac{\pi^2}{45} + \frac{1}{3\pi^2} G_0 + \frac{1}{30\pi^2}(5G_1 + 2G_2)k^2\right]\right\} \\ \Omega_4^{33}(\mathbf{k}) &= \omega_p^4 \left\langle \gamma^{-2}\left(1 - \frac{2V^2}{3c^2}\right)\right\rangle \left\{1 + \frac{k^2}{\kappa^2}(3 + \tfrac{4}{15}\beta_p E_{\text{corr}})\right. \\ &\quad \left. + \frac{\beta_p}{\beta_r^4 n\hbar^3 c^3}\left[\frac{\pi^2}{45} + \frac{1}{3\pi^2} G_0 + \frac{1}{30\pi^2}(5G_1 + G_2)\right]\right\} \end{aligned} \tag{16}$$

where

$$
\begin{aligned}
E_{\text{corr}} &= \frac{n}{2V}\sum_{\mathbf{q}} \frac{4\pi e^2}{q^2}\, g_{\mathbf{q}} \\
G_0 &= \int dx\, x^2 f(x)\, n g_{\mathbf{q}} \\
G_1 &= \int dx\, x^2 f(x) \frac{1}{q}\frac{\partial}{\partial q}\, n g_{\mathbf{q}} \\
G_2 &= \int dx\, x^2 f(x) \frac{\partial^2}{\partial q^2}\, n g_q \\
\kappa^2 &= 4\pi e^2 n \beta_p
\end{aligned}
\tag{17}
$$

It can be seen that relativistic results are different from the corresponding nonrelativistic ones (Kalman and Genga, 1986) by a factor of $\langle\gamma^{-1}(1-V^2/3c^2)\rangle$ for $\Omega_2^{\mu}(\mathbf{k})$ and $\langle\gamma^{-2}(1-2V^2/3c^2)\rangle$ for $\Omega_4^{\mu\nu}(\mathbf{k})$.

I now determine the possible relativistic implication for the dispersion relation of transverse plasma modes. The dispersion relation which determines the behavior of longitudinal plasmons is given by

$$
\varepsilon_{33}(\mathbf{k}\omega) = 1 + \alpha_{33}(\mathbf{k}\omega) \tag{18}
$$

After applying a small perturbation on the dispersion relation, one finds that the plasmon frequency becomes

$$
\omega^2(\mathbf{k}) = \omega_p^2 \left\langle \gamma^{-1}\left(1 - \frac{v^2}{3c^2}\right)\right\rangle \left[1 + C_L(\gamma, \beta_r) + A_L(\gamma, \beta_r)\frac{k^2}{\kappa^2}\right] \tag{19}
$$

where

$$
\begin{aligned}
A_L(\gamma, \beta_r) &= 3 + \tfrac{4}{15}\beta_p E_{\text{corr}} + \frac{1}{30\pi^2}\frac{\beta_p \kappa^2}{\beta_r^4 n \hbar^3 c^3}(5G_1 + G_2) \\
C_L(\gamma, \beta_r) &= \frac{\beta_p}{\beta_r^4 n h^3 c^3}\left(\frac{\pi^2}{9} + \frac{5G_0}{3\pi^2}\right)
\end{aligned}
\tag{20}
$$

For transverse photons, one determines the behavior of the system through the dispersion relation

$$
\varepsilon_{11}(\mathbf{k}\omega) \equiv 1 + \alpha_{11}(\mathbf{k}\omega) = n^2 \tag{21}
$$

This results in the photon frequency of the form

$$\omega^2(\boldsymbol{K}) = \omega_p^2\left\langle \gamma^{-1}\left(1-\frac{V^2}{3c^2}\right)\right\rangle\left[1+c_L(\gamma,\beta_r)+A_\tau(\gamma,\beta_r)\frac{k^2}{\kappa^2}\right]+k^2c^2 \quad (22)$$

where

$$A_\tau = 1-\tfrac{2}{15}\beta_p E_{\text{corr}}+\frac{\beta_p}{30\pi^2}\frac{\kappa^2}{\beta_r^4 n\hbar^3c^3}(5G_1+2G_2) \quad (23)$$

The relativistic longitudinal plasma and transverse photon frequencies are different from the corresponding nonrelativistic ones (Kalman and Genga, 1986) by a factor of $\langle\gamma^{-1}(1-V^2/3c^2)\rangle$.

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